Diffusive teleportation on a quantum dot chain

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We introduce a model of quantum teleportation on a channel built on a quantum dot chain. Quantum dots are coupled through hopping and each dot can accept zero, one or two electrons. Vacuum and double occupation states have the same potential energy, while single occupation states are characterized by a lower potential energy. A single dot initially decoupled from the others is weakly coupled with an external element (Bob), where a pair of electrons has been previously localized. Because of hopping after a suitable time the two dots charge states become maximally entangled. Another chain dot (Alice) is put in an unknown superposition of vacuum and double occupation states, and the other dots are initially empty. The time evolution of the system involves an electron diffusive process. A post selection procedure represented by the detection of charge pairs in a region of the chain equidistant from Alice and Bob, allows, if successful, the reconstruction on the Bob site of the unknown state initially encoded by Alice. The peculiar feature of the model is that the introduction of a trapped magnetic field strongly improves the process efficiency.

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I. INTRODUCTION

In the framework of quantum information [1] a key task is represented by the ability of manipulating a quantum state in order to realize a desired logical operation or to transfer it from one location to another inside a quantum processor. An example of channel proposed for quantum communication is represented by a spin chain [2]. Here the information is encoded in a site by rotating a spin state of an Heisenberg ferromagnet. The presence of an unaligned spin creates a diffusion that allows to reconstruct the information (i.e. the spin state) on a different site with a good fidelity. Such a channel is also useful to transfer spin entanglement, as showed in Ref. [3]. An XY Hamiltonian permits a state transfer over long distances with fidelity 1 [4].

A quantum channel is defined as the evolution of a system from the initial configuration to the final one.

The teleportation [5] is a particular kind of quantum channel which exploits an entangled state shared by sender (Alice) and receiver (Bob), plus local measurements and classical communication.

Experimental implementations of teleportation protocol have been reported in quantum optics [6, 7, 8], NMR [9] and very recently in atom physics [10, 11], and proposals for solid state are currently object of interest [12, 13, 14, 15, 16].

This paper is devoted to the introduction of a model which describes a teleportation on a quantum dot (QD) chain based on a diffusive mechanisms. An unexpected feature of the model is that the success probability of the protocol is enhanced through the use of a trapped magnetic field proposed to improve the efficiency of the scheme.

The paper is articulated as follows. In section II we introduce the teleportation by extending the usual argument to the effect of a time evolution of the system and introducing a mechanism which implies a probabilistic behavior of the process. In section III is discussed an hamiltonian model which behaves as a channel for diffusive teleportation. The presence of a trapped magnetic field is shown to improve the efficiency of the process. Last section will be devoted to conclusions.

II. TELEPORTATION UNDER TIME EVOLUTION

Generally speaking, the teleportation works as follows. An unknown state is encoded on a quantum two level system (qubit) in the Alice's site: $|A\rangle = \alpha\,|A_1\rangle + \beta\,|A_2\rangle$. Bob, being far from Alice, wants to receive the qubit, preserving the information contained thereinto, by exchanging with Alice only classical information. To do it, they need to share an EPR state. To complete the process, Alice performs a measurement operation (the so called Bell measurement) on its own qubit and one component of the entangled state. The result is then transmitted to Bob by way of classical bits,

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and he can choose a proper unitary transformation to apply on the second part of the shared EPR pair, recovering in such a way the unknown state without need to know it.

Initially the system can be though as composed of an Alice's state $|A\rangle$, a Bob's state and an intermediary system state, entangled with the second one:

$$|\Psi\rangle = |A\rangle \otimes |S, B\rangle = (\alpha\,|A_1\rangle + \beta\,|A_2\rangle) \otimes (|S_1, B_1\rangle + |S_2, B_2\rangle) \tag{1}$$

where $|\alpha|^2 + |\beta|^2 = 1$, and the Hamiltonian determining the evolution is $H = H\,(A \otimes S \otimes B)$. Next, we suppose that the time evolution involves just the subsystem $A \otimes S$, while B evolves in a trivial way. Labelling with $|\Phi_k\rangle$ the energy eigenvectors of $A \otimes S$ and with E_k the respective eigenvalues, fixing $\hbar = 1$, we write $|A_i, S_j\rangle_t = \sum_k a_k^{ij} e^{-iE_k t}\,|\Phi_k\rangle$ (with $i, j = 1, 2$) and

$$|\Psi\rangle = \sum_k e^{-iE_k t}\,|\Phi_k\rangle \left[\left(\alpha a_k^{11} + \beta a_k^{21}\right)|B_1\rangle + \left(\alpha a_k^{12} + \beta a_k^{22}\right)|B_2\rangle\right] \tag{2}$$

The coefficients a_k^{ij} are supposed to be known.

By an energy measurement performed on $A \otimes S$ Bob's state collapses in a coherent superposition of $|B_1\rangle$ and $|B_2\rangle$ with weights $\left(\alpha a_k^{11} + \beta a_k^{21}\right)$ and $\left(\alpha a_k^{12} + \beta a_k^{22}\right)$ connected to α and β by a k-dependent transformation. In order to preserve the universality of the protocol the transformation to implement to recover the incoming Alice's state has to be independent from α and β. In general cases this requirement is not satisfied and the teleportation by energy measurement does not work.

The obstacle above illustrated is related to the difficulty of realizing a full Bell measurement, which can be partially solved by a partial Bell measurement [17]. Alternatively, we propose the idea to create a system which automatically excludes some state as output result. As an example, we would write the overall state introduced in Eq. 1 as

$$|\Psi\rangle = \alpha\,|A_1, S_1\rangle\,|B_1\rangle + \beta\,|A_2, S_2\rangle\,|B_2\rangle \tag{3}$$

Moreover, if the system is initially prepared in its ground state, an energy selection can acts in the desired manner allowing to some state to be more probably populated than some other.

When the simplification applies, Eq. 2 reduces to

$$|\Psi\rangle = \sum_k e^{-iE_k t}\left(\alpha a_k^{11}\,|B_1\rangle + \beta a_k^{22}\,|B_2\rangle\right)|\Phi_k\rangle \tag{4}$$

From Eq. 4 we note that the universality of the teleportation can be observed if and only if a_{1k} and a_{2k} have the same modulus and differ only for a phase factor, i.e. $a_{1k} = e^{i\varphi_k} a_{2k}$ for each k. If the Hamiltonian $H\,(AS)$ exhibits a non-degenerate spectrum Alice sends to Bob the information about the measured energy level as classical bit and finally he can do the conditional unitary operation to completely reconstruct the incoming unknown state. Otherwise, if a_{1k} and a_{2k} are connected by a more complicated relation, there is no way to extract some useful information from an energy measurement without knowing α and β and the procedure fails. If the condition $a_{1k} = e^{i\varphi_k} a_{2k}$ is satisfied just for some k we deal with a teleportation protocol characterized by a success probability less than 1.

The situation above described is not the more general. Effectively, to realize an energy measurement could be hard, and the difficulty increases with the number of states implied in the evolution. The measurement can be performed in a different basis, and a favorable choice is often represented by the computational basis.

Writing a generic element $\left|\tilde{\Phi}_l\right\rangle$ of the new set of orthogonal states as combination of energy eigenstates, Eq. 4 becomes

$$|\Psi\rangle = \sum_l \left(\alpha A_{1l}\,(t)\left|\tilde{\Phi}_l\right\rangle|B_1\rangle + \beta A_{2l}\,(t)\left|\tilde{\Phi}_l\right\rangle|B_2\rangle\right) \tag{5}$$

where $A_{il}\,(t) = \sum_k e^{-iE_k t} a_{ik} b_{kl}$ and $b_{kl} = \left\langle\tilde{\Phi}_l|\Phi_k\right\rangle$.

Now the condition to fulfil to deal with a deterministic teleportation is $A_{1l}\,(t) = e^{i\varphi_l} A_{2l}\,(t)$ and the measurement time is then meaningful. Hence, if for certain times this relation is satisfied, it will be sufficient to perform the Bell measurement at the right time.

However, this is not enough to ensure the protocol to be deterministic. In the computational basis the information derived from a measurement is expressed as "yes" or "not" and the space dimension d plays an important role. If only two are the possible populated states, a measurement will be sufficient to identify the right unitary transformation to perform over the Bob's qubit, while this is not possible if $d > 2$. A complete description of a near deterministic, time dependent, teleportation scheme is done elsewhere [16]. Here we are interested to a model which represents an example of probabilistic teleportation.

III. THE MODEL

The model is represented by a chain of localized levels (QDs) with double occupation allowed. Empty and doubly occupied dots are energetically degenerate, while single occupation is characterized by an energy ϵ. Furthermore, we suppose that no electrons are initially present in the chain. The qubit is defined as the coherent superposition of vacuum and double occupation state in a single QD. Being possible also single occupation, we deal with an "open" two level system.

The unknown state to teleport is encoded on a QD (representing Alice) external to the chain as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|\uparrow\downarrow\rangle \tag{6}$$

where $\uparrow$ stays for spin up and $\downarrow$ stays for spin down. A way to obtain this state is represented by the interaction of the dot initially in its vacuum state with a superconductive lead.

Bob is located in another external QD which is locally coupled with an element of the chain.

The interaction between Bob and the chain dot is described by the Hamiltonian

$$H = H_0 + H_I \tag{7}$$

with

$$H_0 = -2\epsilon \sum_{i=1}^{2} n_{i\uparrow} n_{i\downarrow} + \epsilon \sum_{i=1}^{2} \left(n_{i\uparrow} + n_{i\downarrow}\right) \tag{8}$$

and

$$H_I = -w \sum_{\sigma} \left(c^{\dagger}_{1,\sigma} c_{2,\sigma} + h.c.\right) \tag{9}$$

where $c_{i,\sigma}$ $\left(c^{\dagger}_{i,\sigma}\right)$ is the annihilation (creation) operator on dot i (e.g. the dot 1 is Bob, the dot 2 is the chain element) and σ is the spin index.

If an electron pair is prepared on Bob's dot, the incoming state is $|\Phi_1\,(t=0)\rangle = |\uparrow\downarrow, 0\rangle$. The other possible states are $|\Phi_2\,(t=0)\rangle = |0, \uparrow\downarrow\rangle$, $|\Psi_1\,(t=0)\rangle = 2^{-1/2}\left(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle\right)$, and $|\Psi_2\,(t=0)\rangle = 2^{-1/2}\left(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle\right)$.

In the limit $\epsilon >> w$ the state $|\Phi_1\,(t=0)\rangle$ evolves, aside from a global phase factor, in $|\Phi_1\,(t)\rangle = \left[\cos\frac{\omega t}{2}\,|\Phi_1\,(t=0)\rangle + i\sin\frac{\omega t}{2}\,|\Phi_2\,(t=0)\rangle\right] + O(w/\epsilon)$, where $\omega = \epsilon - \sqrt{(\epsilon^2 + 16 w^2)}$. If the tunneling is switched off when $\omega t/2 = \pi/4$ a maximally entangled state is then generated.

After entanglement is created, the whole state describing Alice, the chain and Bob is

$$|\psi\rangle = \frac{1}{2}\left(\alpha\,|0\rangle_A\,|0\rangle_C\,|\uparrow\downarrow\rangle_B + \beta\,|\uparrow\downarrow\rangle_A\,|0\rangle_C\,|\uparrow\downarrow\rangle_B + \alpha\,|0\rangle_A\,|\uparrow\downarrow\rangle_C\,|0\rangle_B + \beta\,|\uparrow\downarrow\rangle_A\,|\uparrow\downarrow\rangle_C\,|0\rangle_B\right) \tag{10}$$

where subscript A denotes Alice'site, B is Bob's site and C is the chain site nearest to Bob.

If Alice's dot is embedded in the chain and a sort of chemical potential is added to the Hamiltonian in order to limit to one the number of electron pairs on the enlarged chain, we obtain

$$|\psi\rangle = \alpha\,|0\rangle_A\,|\uparrow\downarrow\rangle_C\,|0\rangle_B + \beta\,|\uparrow\downarrow\rangle_A\,|0\rangle_C\,|\uparrow\downarrow\rangle_B \tag{11}$$

in analogy with Eq. 3.

If the Hamiltonian introduced in Eq. 7 is now turned on for all the elements in the chain, Bob is excluded, the states $|0\rangle_A\,|\uparrow\downarrow\rangle_C$ and $|\uparrow\downarrow\rangle_A\,|0\rangle_C$ experience a time evolution that consists in a diffusion of the localized pair around the ring.

The diffusion is studied by the introduction of $|\Psi_{l,m}\rangle$, describing the presence of an electron with spin up (down) on the site $l(m)$ and its Fourier transform $\left|\tilde{\Psi}_{k,q}\right\rangle$

$$\left|\tilde{\Psi}_{k,q}\right\rangle = \frac{1}{N}\sum_{l,m} |\Psi_{l,m}\rangle\, e^{ikl} e^{imq} \tag{12}$$

If we suppose that Alice is located in the site labeled with 0 and Bob is close to the m-th site, the initial state is

$$|\Psi\,(t=0)\rangle = \alpha\,|\Psi_{m,m}\,(t=0)\rangle\,|0\rangle_B + \beta\,|\Psi_{0,0}\,(t=0)\rangle\,|\uparrow\downarrow\rangle_B \tag{13}$$

In the complex Laplace space the evolution is given by

$$\begin{aligned}(\omega-\epsilon)\left|\Psi_{l,m}(\omega)\right\rangle = \left|\Psi_{l,m}(t=0)\right\rangle - \\ w\left[\left|\Psi_{l+1,m}(\omega)\right\rangle + \left|\Psi_{l,m-1}(\omega)\right\rangle + \left|\Psi_{l-1,m}(\omega)\right\rangle + \left|\Psi_{l,m+1}(\omega)\right\rangle\right] - \epsilon\left|\Psi_{l,m}(\omega)\right\rangle \delta_{l,m}\end{aligned} \tag{14}$$

which is derived taking into account that states with double occupation on a site have zero potential energy, and from which follows

$$\left|\Psi_{m,m}(\omega)\right\rangle = \frac{1}{N}\sum_{k,q}\frac{e^{-i(k+q)m}\left|\tilde{\Psi}_{k,q}(t=0)\right\rangle - \frac{\epsilon}{N}\sum_l\left|\Psi_{l,l}(\omega)\right\rangle e^{i(l-m)(k+q)}}{\omega-\epsilon+2w(\cos k+\cos q)} \tag{15}$$

If a trapped magnetic field is introduced, because of the Aharonov-Bohm effect [18], the electron bands are shifted by a quantity $\Phi = e\Phi'/\hbar$, being Φ' the flux of the magnetic field and e the electron charge.

The Bell measurement is performed by selecting a site l_0 (from symmetry reasons it will be natural to choose the intermediate dot between 0 and m) and asking to find two charges here localized: the Bob's state is then reduced to

$$|B(t)\rangle = \alpha f_{l_0,m}(t)\,|0\rangle_B + \beta f_{l_0,0}(t)\,|\uparrow\downarrow\rangle_B \tag{16}$$

where $f_{l_0,p}(t) = \langle\Psi_{l_0,l_0}(t=0)|\Psi_{p,p}(t)\rangle$ $(p=0,m)$.

After some algebraic manipulation, whose details are given in appendix A, we find

$$f_{l_0,p}(\omega) = \frac{1}{\epsilon N}\sum_k e^{+i(l_0-p)k}\frac{\Lambda(k,\omega)}{1+\Lambda(k,\omega)} \tag{17}$$

with

$$\Lambda(k,\omega) = \frac{\epsilon}{\sqrt{(\omega-\epsilon)^2-\left(4w\cos\left(\frac{k}{2}-\Phi\right)\right)^2}} \tag{18}$$

or

$$1+\Lambda(k,\omega) = \frac{\epsilon}{\sqrt{(\omega-\epsilon)^2-\left(4w\cos\left(\frac{k}{2}-\Phi\right)\right)^2}+\epsilon}$$

Here we note that the cut deriving from the square root gives a negligible contribute and that a pole is present in $1+\Lambda(k,\omega)$ only if $\epsilon<0$. Then, the two level of qubit encoding represent excited states of the system.

Quite naturally, the projection can be performed on a region of finite extension around l_0. If a gaussian form factor of width σ is used, we get

$$f_{l_0,m}(t) = \frac{1}{N}\sum_k e^{ikl_0}e^{-\frac{k^2\sigma^2}{2}}\frac{e^{i\epsilon t}}{2\pi i}\oint d\omega\frac{e^{i\omega t}\left[\sqrt{\omega^2-8w^2\left[1+\cos(k-2\Phi)\right]}-\epsilon\right]}{\omega^2-8w^2\left[1+\cos(k-2\Phi)\right]-\epsilon^2} \tag{19}$$

and

$$f_{l_0,0}(t) = \frac{1}{N}\sum_k e^{-ikl_0}e^{-\frac{k^2\sigma^2}{2}}\frac{e^{i\epsilon t}}{2\pi i}\int_C d\omega\frac{e^{i\omega t}\left[\sqrt{\omega^2-8w^2\left[1+\cos(k-2\Phi)\right]}-\epsilon\right]}{\omega^2-8w^2\left[1+\cos(k-2\Phi)\right]-\epsilon^2} \tag{20}$$

We note that the gaussian distribution makes relevant only small values of k. The roots of the denominator of the latter equations are

$$\omega_\pm = \pm\sqrt{\epsilon^2+8w^2\left[1+\cos(k-2\Phi)\right]} \tag{21}$$

and, defining $\epsilon_\pm = \epsilon+\omega_\pm$, are expressible as $\epsilon_\pm \simeq \epsilon_\pm(k=0)+\epsilon'_\pm(k=0)\,k+\epsilon''_\pm(k=0)\,\frac{k^2}{2}$ with the symmetry properties $\epsilon''_+(0) = -\epsilon''_-(0)$ and $\epsilon'_+(0) = -\epsilon'_-(0)$. If the trapped field were not present, the linear term in k should be zero, due to the cosine dependence.

By inverse Laplace transform we have

$$\begin{aligned}f\ \ _{l_0,m}(t) = \tfrac{2}{N}\textstyle\sum_k \frac{|\epsilon|}{\epsilon_+-\epsilon_-}\{\exp\left[i\epsilon_+(0)\,t+ik\left[\epsilon'_+(0)\,t-l_0\right]-\frac{k^2\lambda(t)}{2}\right] - \\ -\exp\left[i\epsilon_-(0)\,t+ik\left[-\epsilon'_+(0)\,t-l_0\right]-\frac{k^2\lambda^*(t)}{2}\right]\}\end{aligned} \tag{22}$$

and

$$f_{l_0,0}(t) = \frac{2}{N}\sum_k \frac{|\epsilon|}{\epsilon_+ - \epsilon_-} \left\{ \exp\left[i\epsilon_+(0)t + ik\left[\epsilon'_+(0)t + l_0\right] - \frac{k^2\lambda(t)}{2}\right] - \right.$$
$$\left. - \exp\left[i\epsilon_-(0)t + ik\left[-\epsilon'_+(0)t + l_0\right] - \frac{k^2\lambda^*(t)}{2}\right]\right\} \tag{23}$$

with $\lambda(t) = \sigma^2 + i\epsilon''_+(0)t$.

The linear term in k is responsible of a quick attenuation because of interference of different wavepacket components.

The introduction of the trapped magnetic field eliminates this attenuation: without the field the term $\epsilon'_+(0)$ is not present, and there is no way to bring to zero the rapid diffusion term. On the other hand, in the presence of the field, by choosing a proper time ($t_M = l_0/\epsilon'_+(0)$) for the projective measurement, one of the linear terms in k cancels both in equation 22 and in equation 23, giving rise to a coherent propagation of the two electron wavepacket. The presence of the term proportional to k^2 implies a broadening of the gaussian wave function which can be optimized by a suitable choice of the intensity of the magnetic field. Intuitively, the two exponentials in each of latter equations represent counterpropagating terms. The magnetic field has the effect to keep the coherence of peaks. If usually two peaks starting from different points are subject to a inessential global effect on the phase, the peculiar structure above derived allows to superpose two peaks starting form opposite point in the middle of the chain.

After the projection Bob's state is then

$$\begin{aligned} |B(t_M)\rangle &= \alpha\frac{2}{N}\sum_k \frac{|\epsilon|}{\epsilon_+ - \epsilon_-}\exp\left(i\epsilon_+(0)t_M - \frac{k^2\lambda(t_M)}{2}\right)|0\rangle_B - \\ &- \beta\frac{2}{N}\sum_k \frac{|\epsilon|}{\epsilon_+ - \epsilon_-}\left[\exp\left(i\epsilon_-(0)t_M - \frac{k^2\lambda^*(t_M)}{2}\right)\right]|\uparrow\downarrow\rangle_B \end{aligned} \tag{24}$$

Performing the sum over k imposing $k = 0$ in the difference $(\epsilon_+ - \epsilon_-)$ we get

$$|B(t_M)\rangle = e^{i[(\epsilon-\epsilon_0)\ t_M + \theta(t_M)]}\,|f(t_M)|\left\{\alpha e^{i\left[2\epsilon_0\ t_M - 2\theta(t_M) + \frac{\pi}{2}\right]}|0\rangle_B + \beta|\uparrow\downarrow\rangle_B\right\} \tag{25}$$

where

$$f(t_M) \simeq \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda^*(t_M)}\frac{|\epsilon|}{\epsilon_0} \tag{26}$$

with $f(t_M) = |f(t_M)|\,e^{i\theta(t_M)}$ and $\epsilon_0 = 2\sqrt{\epsilon^2 + 8w^2(1+\cos 2\Phi)}$.

From Eq. 25 we learn that the state transferred to Bob is the same initially encoded by Alice, apart from an inessential global phase factor, a relative phase factor whose effects are eliminable by a deterministic unitary operation, known *a priori*, different from the classically driven unitary rotation in the standard teleportation, and a term $(|f(t_M)|)$ which creates attenuation: the scheme efficiency is then $|f(t_M)|^2$.

Writing explicitly

$$|f(t_M)| = \frac{1}{\sqrt{2\pi}}\frac{|\epsilon|}{\epsilon_0}\frac{1}{\sqrt{\sigma^4 + t_M^2\frac{16w^4}{\epsilon_0^2}\cos^2 2\Phi}} \tag{27}$$

we note that the minimum attenuation is reached when $\Phi = \pi/4$ which implies $t_M = l_0\epsilon_0/4w^2$.

In the usual teleportation, after the Bell measurement, a classical communication related to a unitary transformation is needed to warrant against superluminar information transfer. Here, apparently, the transfer is realized independently from the unitary operation. Actually, due to the probabilistic nature of the process, Bob has to know, through a classical channel, if Alice's measurement were successful, and only in case of affirmative answer he is sure to be in possess of the right unknown quantum state.

If the constraint of only one electron pair propagating on the chain is relaxed, the probability amplitude of observing a pair in the middle point is modified by the contribution of configurations where two other spins are somewhere in the chain (the last term in Eq. 10). A peculiar feature of the model is that this contribution is not enhanced by the magnetic field and its effects on the scheme is negligible. In appendix B we shall show analytically this argument.

IV. CONCLUSIONS

We have studied the teleportation protocol introducing the possibility that some subcomponent is subject to time evolution. A model which fulfils this characteristic has been introduced by means of a chain of interacting quantum dots. To improve the protocol efficiency, a trapped magnetic field has been introduced. We observed that a coherent, although attenuated, propagation from Alice and Bob of two counterpropagating wavepackets, plus a postselection measurement which determines the presence of a particle pair in the middle point at a selected time permits to realize the teleportation. The model proposed is an "open quantum system" with the qubit defined on two excited states. A natural limit is represented by unavoidable attenuation due to diffusion and fragility with respect to decoherence eventually due to interaction with an external phonon bath, since the mechanism works exploiting local excited states on each QD.

APPENDIX A: DERIVATION OF COEFFICIENTS IN EQ. 16

We have to calculate the coefficient

$$f_{l,m}(\omega) = \langle \Psi_{l,l}(t=0) | \Psi_{m,m}(\omega) \rangle \tag{A-1}$$

where l, m are now two generic sites, starting from Eq. 15, which we recall for convenience:

$$|\Psi_{m,m}(\omega)\rangle = \frac{1}{N} \sum_{k,q} \frac{e^{-i(k+q)m} \left| \tilde{\Psi}_{k,q}(t=0) \right\rangle - \frac{\epsilon}{N} \sum_l |\Psi_{l,l}(\omega)\rangle e^{i(l-m)(k+q)}}{\omega - \epsilon + 2w(\cos k + \cos q)} \tag{A-2}$$

Noting that the inner product $\left\langle \Psi_{l,l}(t=0) | \tilde{\Psi}_{k,q}(t=0) \right\rangle$ is equal to $e^{+i(k+q)l}/N$, Eq. A-2 becomes

$$f_{l,m}(\omega) = \frac{1}{N} \sum_{k,q} \frac{1}{\omega - \epsilon + 2w(\cos k + \cos q)} \left[\frac{1}{N} e^{-i(m-l)(k+q)} - \frac{\epsilon}{N} \sum_{l'} f_{l,l'}(\omega) e^{i\left(l'-m\right)(k+q)} \right] \tag{A-3}$$

or

$$f_{l,m}(\omega) = \frac{1}{N} \sum_{k,q} \frac{1}{\omega - \epsilon + 2w[\cos(k-q) + \cos q]} \left[\frac{1}{N} e^{-i(m-l)k} - \frac{\epsilon}{N} \sum_{l'} f_{l,l'}(\omega) e^{i\left(l'-m\right)k} \right] \tag{A-4}$$

By defining the quantity

$$f_l(k,\omega) = \frac{1}{\sqrt{N}} \sum_{l'} f_{l,l'}(\omega) e^{ikl'} \tag{A-5}$$

we obtain the following identity:

$$f_l(k,\omega) = \frac{1}{\sqrt{N}} \sum_q \frac{1}{\omega - \epsilon + 2w[\cos(k-q) + \cos q]} \left[\frac{1}{N} e^{+ilk} - \frac{\epsilon}{\sqrt{N}} f_l(k,\omega) \right] \tag{A-6}$$

which can be rewritten as

$$f_l(k,\omega) = \frac{1}{\epsilon\sqrt{N}} e^{+ilk} \frac{\Lambda(k,\omega)}{1 + \Lambda(k,\omega)} \tag{A-7}$$

where

$$\Lambda(k,\omega) = \frac{\epsilon}{N} \sum_q \frac{1}{\omega - \epsilon + 4w\cos\left(q - \frac{k}{2}\right)\cos\frac{k}{2}} \tag{A-8}$$

After substitution, we get

$$f_{l,m}(\omega) = \frac{1}{\epsilon N} \sum_k e^{+i(l-m)k} \frac{\Lambda(k,\omega)}{1 + \Lambda(k,\omega)} \tag{A-9}$$

Equation 18 is derived performing the sum over q in the continuous limit.

APPENDIX B: TIME EVOLUTION OF TWO ELECTRON PAIRS

In this appendix we discuss the case of time evolution of last term in Eq. 10. The diffusion arises from an initial state where the first electron pair is localized in the site l and the second is localized in the site m. The system state is described through its Laplace transform $|\Psi_{l,m;n,p}(\omega)\rangle$ which is subject to

$$\begin{aligned}(\omega-2\epsilon)|\Psi_{l,m;n,p}(\omega)\rangle = |\Psi_{l,m;n,p}(t=0)\rangle - 2\epsilon|\Psi_{l,m;n,p}(\omega)\rangle(\delta_{l,n}\delta_{m,p}+\delta_{l,p}\delta_{m,n}) - \\ -\epsilon|\Psi_{l,m;n,p}(\omega)\rangle(\delta_{l,n}+\delta_{l,p}+\delta_{m,n}+\delta_{m,p}) - \\ -w[|\Psi_{l-1,m;n,p}(\omega)\rangle + |\Psi_{l+1,m;n,p}(\omega)\rangle + |\Psi_{l,m-1;n,p}(\omega)\rangle + |\Psi_{l,m+1;n,p}(\omega)\rangle + \\ +|\Psi_{l,m;n-1,p}(\omega)\rangle + |\Psi_{l,m;n+1,p}(\omega)\rangle + |\Psi_{l,m;n,p-1}(\omega)\rangle + |\Psi_{l,m;n,p+1}(\omega)\rangle]\end{aligned} \tag{B-1}$$

obtained taking into account that states with different number of paired electrons give a different contribution to the potential energy.

By the introduction of Fourier transform, defined as

$$|\Psi_{k,q;r,s}(\omega)\rangle = \frac{1}{N^2}\sum_{l,m,n,p} e^{ikl}e^{iqm}e^{irn}e^{isp}|\Psi_{l,m;n,p}(\omega)\rangle \tag{B-2}$$

we are able to write

$$\begin{aligned}\left|\tilde{\Psi}_{k,q;r,s}(\omega)\right\rangle = \frac{1}{\omega-\epsilon(k,q,r,s)}* \\ [\left|\tilde{\Psi}_{k,q;r,s}(t=0)\right\rangle - \frac{\epsilon}{N^2}\sum_{l,m,p}|\Psi_{l,m;l,p}(\omega)\rangle e^{i(k+r)l}e^{iqm}e^{isp} - \frac{\epsilon}{N^2}\sum_{l,m,n}|\Psi_{l,m;n,l}(\omega)\rangle e^{i(k+s)l}e^{iqm}e^{irn} - \\ \frac{\epsilon}{N^2}\sum_{l,m,p}|\Psi_{l,m;m,p}(\omega)\rangle e^{ikl}e^{i(q+r)m}e^{isp} - \frac{\epsilon}{N^2}\sum_{l,m,n}|\Psi_{l,m;n,m}(\omega)\rangle e^{ikl}e^{i(qb+s)m}e^{irn} - \\ \frac{2\epsilon}{N^2}\sum_{l,m}|\Psi_{l,m;l,m}(\omega)\rangle e^{i(k+r)l}e^{i(q+s)m} - \frac{2\epsilon}{N^2}\sum_{l,m}|\Psi_{l,m;m,l}(\omega)\rangle e^{i(k+s)l}e^{i(q+r)m}]\end{aligned} \tag{B-3}$$

having introduced $\epsilon(k,q,r,s) = [2\epsilon - 2w(\cos k + \cos q + \cos r + \cos s)]$.

Then, the projection on the site l_0 of two pairs starting from l and m, which we call $f(l,m,l_0)$, is related to the inner product $\langle\Psi_{l_0,m';l_0,p'}|\Psi_{l,m;l,m}(\omega)\rangle$ (being m' and p' the sites occupied by the other two electrons).

$$f(l,m,l_0;\omega) = \sum_{m',p'}\langle\Psi_{l_0,m';l_0,p'}|\Psi_{l,m;l,m}(\omega)\rangle \tag{B-4}$$

By applying Fourier transform we obtain

$$f(l,m,l_0;\omega) = \frac{1}{N^4}\sum_{m',p'}\sum_{k',q',r',s'}\sum_{k,q,r,s} e^{-il_0(k'+r')}e^{-i(q'm'+s'p')}e^{il(k+r)}e^{im(q+s)}\left\langle\tilde{\Psi}_{k',q';r',s'}|\tilde{\Psi}_{k,q;r,s}(\omega)\right\rangle \tag{B-5}$$

that, performing the sums over m' and p', reduces to

$$f(l,m,l_0;\omega) = \frac{1}{N^2}\sum_{k',r',k,q,r,s}\left[e^{-il_0(k'+r')}e^{il(k+r)}e^{i(mq+ps)}\left\langle\tilde{\Psi}_{k',0;r',0}|\tilde{\Psi}_{k,q;r,s}(\omega)\right\rangle\right] \tag{B-6}$$

Defining

$$\left|\tilde{\Psi}_1(k,q,s;\omega)\right\rangle = \frac{1}{\sqrt{N}}\sum_r\left|\tilde{\Psi}_{k-r,q;r,s}(\omega)\right\rangle \tag{B-7}$$

$$\left|\tilde{\Psi}_2(k,q,r;\omega)\right\rangle = \frac{1}{\sqrt{N}}\sum_s\left|\tilde{\Psi}_{k-s,q;r,s}(\omega)\right\rangle \tag{B-8}$$

$$\left|\tilde{\Psi}_3(k,q,s;\omega)\right\rangle = \frac{1}{\sqrt{N}}\sum_r\left|\tilde{\Psi}_{k,q-r;r,s}(\omega)\right\rangle \tag{B-9}$$

$$\left|\tilde{\Psi}_4\left(k,q,r;\omega\right)\right\rangle = \frac{1}{\sqrt{N}}\sum_{s}\left|\tilde{\Psi}_{k,q-s;r,s}\left(\omega\right)\right\rangle \tag{B-10}$$

$$\left|\tilde{\Psi}_5\left(k,q;\omega\right)\right\rangle = \frac{1}{N}\sum_{r,s}\left|\tilde{\Psi}_{k-s,q-r,r,s}\left(\omega\right)\right\rangle \tag{B-11}$$

$$\left|\tilde{\Psi}_6\left(k,q;\omega\right)\right\rangle = \frac{1}{N}\sum_{r,s}\left|\tilde{\Psi}_{k-r,q-s,r,s}\left(\omega\right)\right\rangle \tag{B-12}$$

we find that

$$\begin{aligned} f\left(l,m,l_0;\omega\right) = \frac{1}{N}\sum_{k',k,q}\frac{1}{1+2\Pi\left(k,q\right)}e^{-il_0k'}e^{ilk}e^{imq}\frac{1}{N}\sum_{r,s}\frac{1}{\omega-\epsilon\left(k-s,q-r,r,s\right)}\left\langle\tilde{\Psi}_{k',0;r',0}\left(t=0\right)\right| \\ \{\left|\tilde{\Psi}_{k-s,q-r;r,s}\left(t=0\right)\right\rangle - \frac{\epsilon}{\sqrt{N}}\left|\tilde{\Psi}_1\left(k+r-s,q-r,s;\omega\right)\right\rangle - \frac{\epsilon}{\sqrt{N}}\left|\tilde{\Psi}_2\left(k,q-r,r;\omega\right)\right\rangle - \\ \frac{\epsilon}{\sqrt{N}}\left|\tilde{\Psi}_3\left(k-s,q,s;\omega\right)\right\rangle - \frac{\epsilon}{\sqrt{N}}\left|\tilde{\Psi}_4\left(k-s,q-r+s,r;\omega\right)\right\rangle - \frac{2\epsilon}{N}\left|\tilde{\Psi}_6\left(k-s+r,q-r+s;\omega\right)\right\rangle\} \end{aligned} \tag{B-13}$$

where

$$\Pi\left(k,q\right) = \frac{1}{N^2}\sum_{r,s}\frac{1}{\omega-\epsilon\left(k-s,q-r,r,s\right)} \tag{B-14}$$

Eq. B-13 has now to be integrated in the complex plane. We note that the unique contribution giving rise to a pure pole is the inner product $\left\langle\tilde{\Psi}_{k',0;r',0}\left(t=0\right)|\tilde{\Psi}_{k-s,q-r;r,s}\left(t=0\right)\right\rangle$, because all other terms present an integration over the pole and can be, in a perturbative approach, neglected.

Hence, applying conservation rules due to index matching,

$$f\left(l,m,l_0;\omega\right) = \frac{1}{N^2}\sum_{k',k,q}\frac{e^{i(l-l_0)k}e^{i(m-l_0)q}}{1+2\Pi\left(k,q\right)}\frac{1}{\omega-\epsilon\left(k,0,q,0\right)} \tag{B-15}$$

Limiting ourselves to small values of q and k, the pole is in

$$\omega_0 = 2\epsilon + 2w\left(k^2+q^2\right) \tag{B-16}$$

Then

$$f\left(l,m,l_0;t\right) = \frac{1}{N^2}\sum_{k,q}\frac{e^{idk}e^{-idq}e^{i\omega_0 t}}{\left[1+2\Pi\left(k,q,\omega_0\right)\right]} \tag{B-17}$$

where $d = l - l_0 = l_0 - m$.

The effect of the trapped magnetic field manifests itself giving for the pole $\omega_0 = 2\epsilon + 2w\left(2\Phi^2+2\Phi k+k^2+2\Phi q+q^2\right)$.

Then the linear part in q and k is $\exp i\left[d\left(k-q\right)+4w\Phi\left(k+q\right)t\right]$. A proper choice of time for projecting the state will permit only to eliminate one of two terms responsible for rapid diffusion, and the probability of finding two electrons on the site l_0 will be negligible.

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